

Two deformations of a fermionic solution to pentagon equation

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Abstract

Two novel fermionic — expressed in terms of Grassmann–Berezin calculus of anticommuting variables — solutions of pentagon equation are proposed, both being deformations of the known solution related to the affine group.

1 Introduction

By pentagon equation we understand an algebraic relation which can be said to correspond naturally to a Pachner move $2 \rightarrow 3$ — an elementary rebuilding of a 3-manifold triangulation, which replaces two tetrahedra 1234 and 1235 with three tetrahedra 1245, 1345 and 2345 occupying the same place in the manifold. If some quantities satisfy this relation, we say that a solution to pentagon equation has been found.

Many interesting solutions of pentagon equation are related to quantum invariants of 3-manifolds, and these were certainly a source of inspiration in our search. We are interested, however, in contrast to the usual approach, in solutions expressed in terms of Grassmann–Berezin calculus of anticommuting variables.

The particular kind of pentagon equation dealt with in this paper is (5), while its solutions are (4), (11) and (13). The already known solution is (4); there exists a conceptual proof of its validity and even a whole theory relating it to Reidemeister torsion of some unusual chain complexes associated with an affine group, see [2]. It can also be expressed naturally as a fermionic Gaussian integral, we do it here in formula (8).

Our new solutions in this paper are (11) and (13), and they look even more unusual. Although we manage to represent them, too, in a form of Gaussian integral, their possible relations to Reidemeister torsions look obscure, and our proofs of their validity consist in direct calculations. Happily, these proofs are much simplified due to the fact that both (11) and (13) appear as “perturbations” of (4), namely by adding a new term of higher or lower degree, respectively, in anticommuting variables.

Below, in Section 2 we recall the necessary things about the calculus of anticommuting variables; in Section 3 we recall our “old” pentagon equation solution. Then we present our new solutions in Section 4, write out a Gaussian integral form for them in Section 5 and finish with a brief discussion in Section 6.

2 Grassmann algebras and Berezin integral

A *Grassmann algebra* over a field \mathbb{F} — for which we can take in this paper any field of characteristic $\neq 2$ — is an associative algebra with unity, having generators a_i and relations

$$a_i a_j = -a_j a_i.$$

As this implies for $i = j$ that $a_i^2 = 0$, any element of a Grassmann algebra is a polynomial of degree ≤ 1 in each a_i . For a given Grassmann monomial, by its degree we understand its total degree in all Grassmann variables; if an element of Grassmann algebra includes only monomials of odd degrees, it is called odd; if it includes only monomials of even degrees, it is called even.

The *exponent* is defined by the standard Taylor series. For example,

$$\exp(a_1 a_2) = 1 + a_1 a_2.$$

If φ_1 and φ_2 are two even elements, then

$$\exp(\varphi_1) \exp(\varphi_2) = \exp(\varphi_1 + \varphi_2). \quad (1)$$

The *Berezin integral* [1] is an \mathbb{F} -linear operator in a Grassmann algebra defined by equalities

$$\int da_i = 0, \quad \int a_i da_i = 1, \quad \int gh da_i = g \int h da_i, \quad (2)$$

if g does not depend on a_i (that is, generator a_i does not enter the expression for g); multiple integral is understood as iterated one, according to the following model:

$$\iint ab db da = \int a \left(\int b db \right) da = 1. \quad (3)$$

3 Solution of pentagon equation related to affine group

3.1 Tetrahedron weight

We ascribe a *coordinate* $\zeta_i \in \mathbb{F}$ to every vertex $i = 1, \dots, 5$ of tetrahedra taking part in the move $2 \rightarrow 3$, see the first paragraph of Section 1. It will be also convenient to use the notation

$$\zeta_{ij} \stackrel{\text{def}}{=} \zeta_i - \zeta_j.$$

Following paper [2]¹, we attach anticommuting Grassmann generators to *unoriented* 2-faces, such as $a_{123} = a_{132} = \dots = a_{321}$, and introduce the following function of coordinates and these generators — the fermionic “Boltzmann weight” of a tetrahedron. To avoid bulky notations, we write it out for tetrahedron 1234; for another tetrahedron $i_1 i_2 i_3 i_4$ just change $k \mapsto i_k$, $k = 1, \dots, 4$:

$$\begin{aligned} \mathbf{f}_{1234} &= \frac{1}{\zeta_{34}} (\zeta_{23} a_{123} - \zeta_{24} a_{124} + \zeta_{34} a_{134}) (\zeta_{13} a_{123} - \zeta_{14} a_{124} + \zeta_{34} a_{234}) \\ &= \zeta_{12} a_{123} a_{124} - \zeta_{13} a_{123} a_{134} + \zeta_{14} a_{124} a_{134} \\ &\quad + \zeta_{23} a_{123} a_{234} - \zeta_{24} a_{124} a_{234} + \zeta_{34} a_{134} a_{234}. \end{aligned} \quad (4)$$

Note that \mathbf{f}_{1234} belongs to an oriented tetrahedron 1234, that is, it changes its sign under a change of orientation.

The weight \mathbf{f}_{1234} is related to the group $\text{Aff}(\mathbb{F})$, i.e., the group of transformations of the form $x \mapsto xa + b$, but we do not explain it here, referring the reader to our paper [2].

3.2 The pentagon equation

As is known from [2], the following pentagon equation² holds for the \mathbf{f} ’s defined by (4):

$$\int \mathbf{f}_{1234} \mathbf{f}_{1235} da_{123} = -\frac{1}{\zeta_{45}} \iiint \mathbf{f}_{1245} \mathbf{f}_{2345} \mathbf{f}_{1345} da_{345} da_{245} da_{145}. \quad (5)$$

See also Subsection 3.4 below for some explanation of this.

3.3 Relation to exponentials of bilinear forms

Associate with tetrahedron 1234 the following matrix (which is to be compared with the expression between two equality signs in (4)):

$$A_{1234} = \begin{pmatrix} \zeta_{23} & -\zeta_{24} & \zeta_{34} & 0 \\ \zeta_{13}/\zeta_{34} & -\zeta_{14}/\zeta_{34} & 0 & 1 \end{pmatrix} \quad (6)$$

and also two more Grassmann generators $b_{1234}^{(1)}$ and $b_{1234}^{(2)}$. Consider the following bilinear form of Grassmann variables:

$$\Phi_{1234} = \begin{pmatrix} b_{1234}^{(1)} & b_{1234}^{(2)} \end{pmatrix} A_{1234} \begin{pmatrix} a_{123} \\ a_{124} \\ a_{134} \\ a_{234} \end{pmatrix}. \quad (7)$$

¹In this paper, we deal only with the “scalar” case of [2], not going into the more complicated matrix case.

²In formula [2, (3)], the convention (3) about the order of multiple integration was adopted. So, [2, (3)] coincides essentially with our formula (5), we only interchanged $da_{145} \leftrightarrow da_{345}$ and wrote the minus sign arising from this. Then, however, there goes a slight confusion in that paper, because, starting from [2, Section 4], a different convention was adopted.

Then it can be seen directly using (2) that the following Gaussian integral representation holds:

$$\mathbf{f}_{1234} = \iint \exp \Phi \, db_{1234}^{(1)} \, db_{1234}^{(2)}. \quad (8)$$

Combining this with the property (1), we see that both sides in (5) can be expressed as multiple (five-fold in the l.h.s. and nine-fold in the r.h.s.) integrals of bilinear forms.

3.4 Some explicit expressions

As this paper is about direct calculations, it makes sense to write out here the matrices of bilinear forms corresponding to the l.h.s. and r.h.s. of (5). The building blocks for them are copies of matrix (6).

The matrix for l.h.s. is

$$\begin{pmatrix} \zeta_{23} & -\zeta_{24} & 0 & \zeta_{34} & 0 & 0 & 0 \\ \zeta_{13}/\zeta_{34} & -\zeta_{14}/\zeta_{34} & 0 & 0 & 0 & 1 & 0 \\ \zeta_{23} & 0 & -\zeta_{25} & 0 & \zeta_{35} & 0 & 0 \\ \zeta_{13}/\zeta_{35} & 0 & -\zeta_{15}/\zeta_{35} & 0 & 0 & 0 & 1 \end{pmatrix}; \quad (9)$$

the rows correspond to $b_{1234}^{(1)}$, $b_{1234}^{(2)}$, $b_{1235}^{(1)}$, and $b_{1235}^{(2)}$; the columns correspond to a_{123} , a_{124} , a_{125} , a_{134} , a_{135} , a_{234} , and a_{235} .

The matrix for r.h.s. is

$$\begin{pmatrix} \zeta_{24} & -\zeta_{25} & 0 & 0 & \zeta_{45} & 0 & 0 & 0 & 0 \\ \zeta_{14}/\zeta_{45} & -\zeta_{15}/\zeta_{45} & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \zeta_{34} & -\zeta_{35} & \zeta_{45} & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta_{14}/\zeta_{45} & -\zeta_{15}/\zeta_{45} & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \zeta_{34} & -\zeta_{35} & \zeta_{45} & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta_{24}/\zeta_{45} & -\zeta_{25}/\zeta_{45} & 0 & 1 \end{pmatrix}; \quad (10)$$

the rows correspond to $b_{1245}^{(1)}$, $b_{1245}^{(2)}$, $b_{1345}^{(1)}$, $b_{1345}^{(2)}$, $b_{2345}^{(1)}$, and $b_{2345}^{(2)}$; the columns correspond to a_{124} , a_{125} , a_{134} , a_{135} , a_{145} , a_{234} , a_{235} , a_{245} , and a_{345} .

The coefficient at every Grassmann monomial in a bilinear form is the minor of its matrix standing in the intersection of the rows and columns corresponding to the variables in that monomial. This reduces the proof of (5) to comparing minors of matrices (9) and (10); such minor must include all rows of the corresponding matrix (because all the b 's must be integrated out) and those columns corresponding to inner faces (for the same reason; the inner faces are, of course, 123 in the l.h.s., and 145, 245, and 345 in the r.h.s.); other columns must correspond to the same a 's in (9) and (10). Also, the signs must be taken into account appearing when we bring a variable to the right in order to integrate it out, as well as the factor $(-1/\zeta_{45})$ in (5).

Fortunately, there exists a theory saving us from actually doing all these calculations, because of a proportionality of the mentioned minors; this is explained in the proof of Theorem 3 in paper [2]. In fact, just one pair of minors must be compared.

4 New solutions

4.1 Solution with term of degree 4

We add one more term — found by method of free search and trial — to (4):

$$\mathbf{g}_{1234} \stackrel{\text{def}}{=} \mathbf{f}_{1234} + \epsilon_{1234} \lambda c_{1234} a_{123} a_{124} a_{134} a_{234}, \quad (11)$$

and similarly, with substitution $k \mapsto i_k$, for any tetrahedron $i_1 i_2 i_3 i_4$. In (11),

$$c_{1234} = \prod_{1 \leq i < j \leq 4} \zeta_{ij},$$

λ is an overall parameter, and ϵ_{1234} is simply the unity, but in general $\epsilon_{i_1 i_2 i_3 i_4} = \pm 1$: if the tetrahedron orientation determined by the order of vertices i_1, i_2, i_3, i_4 is *consistent* with the orientation 1, 2, 3, 4 for tetrahedron 1234, then it is 1, otherwise -1 . Even more directly: $\epsilon_{1235} = -1$, $\epsilon_{1245} = -1$, $\epsilon_{1345} = 1$, and $\epsilon_{2345} = -1$.

Theorem 1. *The \mathbf{g} 's defined by (11) satisfy the same pentagon equation as the \mathbf{f} 's, i.e.,*

$$\int \mathbf{g}_{1234} \mathbf{g}_{1235} da_{123} = -\frac{1}{\zeta_{45}} \iiint \mathbf{g}_{1245} \mathbf{g}_{2345} \mathbf{g}_{1345} da_{345} da_{245} da_{145}. \quad (12)$$

Sketch of the proof. The only known to us proof of Theorem 1 consists in direct calculations. These are simplified by

- (i) the fact that the \mathbf{f} 's already satisfy (5),
- (ii) the fact that $a^2 = 0$ for a Grassmann generator a , and
- (iii) the symmetries of (12): it transforms into itself under any permutation of vertices 1, 2, 3, as well as 4, 5.

It follows from (i) that all monomials of degree 3 in the l.h.s. and r.h.s. of (12) are already the same. Due to (ii), only monomials of degree 5 remain to be checked, and (iii) makes it enough to check the coefficients at just one monomial of degree 5 in both sides of (12), for instance, the factors at $a_{124} a_{125} a_{134} a_{135} a_{235}$. This has been actually done first using paper and pencil and then double-checked using GAP computer algebra system [3]. \square

4.2 Solution with term of degree 0

There is also a somewhat similar but simpler solution of pentagon equation:

$$\mathbf{h}_{1234} \stackrel{\text{def}}{=} \mathbf{f}_{1234} + \epsilon_{1234} \mu, \quad (13)$$

and similarly for other tetrahedra. Here μ is an overall parameter, and $\epsilon_{i_1 i_2 i_3 i_4}$ has the same meaning as in Subsection 4.1.

Theorem 2. *The \mathbf{h} 's defined by (13) satisfy the same pentagon equation as the \mathbf{f} 's and \mathbf{g} 's, namely,*

$$\int \mathbf{h}_{1234} \mathbf{h}_{1235} da_{123} = -\frac{1}{\zeta_{45}} \iiint \mathbf{h}_{1245} \mathbf{h}_{2345} \mathbf{h}_{1345} da_{345} da_{245} da_{145}. \quad (14)$$

Sketch of the proof. Again, the only known to us proof of Theorem 2 consists in direct calculations. The difference with Theorem 1 is that here we must check a monomial of degree 1, instead of 5. \square

5 Representing new solutions as Gaussian integrals

5.1 Gaussian integral for \mathbf{g}

It is not difficult to see directly that our solution (11) can be written in the following Gaussian integral form: replace Φ_{1234} given by (7) with

$$\Gamma_{1234} = \Phi_{1234} + \epsilon_{1234} \lambda_{1234} \zeta_{13} \zeta_{14} \zeta_{23} \zeta_{24} \zeta_{34} a_{134} a_{234},$$

then the analogue of (8) holds:

$$\mathbf{g}_{1234} = \iint \exp \Gamma db_{1234}^{(1)} db_{1234}^{(2)}.$$

5.2 Gaussian integral for \mathbf{h}

Neither is difficult to bring (13) to the Gaussian form: replace Φ_{1234} given by (7) with

$$\Psi_{1234} = \Phi_{1234} + b_{1234}^{(2)} b_{1234}^{(1)},$$

then

$$\mathbf{h}_{1234} = \iint \exp \Psi db_{1234}^{(1)} db_{1234}^{(2)}.$$

5.3 Γ and Ψ not as simple as Φ

One big new feature of forms Γ and Ψ , compared to Φ , is that neither Γ nor Ψ is any longer a form linear, separately, in a 's belonging to 2-faces, on one hand, and b 's belonging to tetrahedra, on the other hand. This makes it problematic to associate with Γ or Ψ , at least in a direct way, a matrix whose copies could be used, first, as building blocks for a larger matrix (like, for a simple instance, (9) or (10)), and then include this larger matrix in a sequence of matrices forming a chain complex. Recall that in [2] and our other papers, the Reidemeister torsion of a complex built in such way was used to construct manifold invariants.

6 Discussion

Here are some concluding remarks:

- The most intriguing thing about our solutions (11) and (13) is that their “mother solution” (4) has a four-dimensional generalization [4] and, in fact, generalizes to any manifold dimension [5]. So, it may make sense to search for higher dimensional generalizations of (11) and (13) as well. This search may be started with *infinitesimal* perturbations of the known solutions: if they exist, this will be already of great interest.
- As we already mentioned, the solution (4) is known [2] to be closely related to the group $\text{Aff}(\mathbb{F})$. At this moment, it is unclear whether this relation is conserved for our new solutions, or maybe $\text{Aff}(\mathbb{F})$ should be replaced by another algebraic object.
- Also, Subsection 5.3 suggests that some generalization of Reidemeister torsion may be needed.
- Of course, the behavior of our solutions with respect to Pachner moves $1 \rightarrow 4$ (a tetrahedron is divided in four, so that a new vertex appears within it) deserves close attention. After obtaining necessary formulas, we can look at what kind of manifold invariants this brings about.
- We could not (as yet?) unite (11) and (13) somehow into one “composite” solution.

References

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